Characterization of Multiple Variable Linear Systems from Random Inputs

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The time constant of a continuous stirred-tank reactor and the transfer function of a simulated second-order system with two random inputs were determined. In the first case the experimental result agreed within 11% of the known value; in the second case the results agreed with those predicted from a theoretical analysis. To determine the transfer matrix of a fifth-order system an impractical amount of medium speed computer time was required; the results were not considered satisfactory. However the length of calculation compares favorably with other methods of total identification for processes which have extraneous noise associated with them. Because of this the application of the present technique in adaptive control systems appears to be favorable.

In the design of process control systems it is necessary that the parameters which characterize the dynamic behavior of the system be known. For conventional feedback control systems this information must be obtained prior to the design of the control system. In an adaptive control system the significant characteristics of the system are automatically measured while the process is in operation.

One method of determining the process characteristics of a linear system is to force the input of the system with a special function, such as a sine wave, an impulse, or a step, and measure the response of the system to this function. Under certain circumstances it is not possible to force a system with such a function or to measure with accuracy the response of the system. For these cases it is possible to apply statistical techniques to analyze the relationship between the input and output of the linear process and thus to determine its process characteristics. If a randomly varying signal is applied to the input of the system, the impulse response function may be obtained by covariance measurements of the input and output; furthermore, the transfer function may be found from spectral density measurements.

Several papers have been presented which discuss the use of statistical methods to study the response of chemical reactor systems. Cowley (*3*) presented two correlation techniques to aid in the frequency response analysis of systems in which the output sine wave was obscured by process noise. These techniques were applied to the determination of the response of a temperature process. One of the methods has also been used in the frequency response analysis of fluidized beds (*19*).

Aris and Amundson (*1*) have derived equations which describe the output of a continuous flow stirred-tank reactor that has random inputs of concentration, or temperature, or flow rate. The same continuous stirred-tank reactor system was studied by Homan and Tierney (*12*) using a simulated model of the stirred-tank reactor and a finite length of a random input signal. This study used a random step input and power spectral density methods. The effects that the sampling interval, the maximum lag, and the total run length have on the estimate of the transfer function were studied. It was found in this study that at least one to two thousand sample points were needed to obtain estimates of the transfer function which are satisfactory for most engineering purposes.

More recently Goodman et al. (*9*) have carried out an experiment with a heat transfer process in which the transfer function was determined with a stationary random input and 750 data points. The method of estimation was again based on power spectral density measurements. They also made runs on a simulated nonlinear reactor. The results indicated fair agreement for the linear heat transfer process (although extraneous noise affected the results at low frequency). For the nonlinear system nothing more than a rather gross qualitative picture could be obtained.

Another recent analysis of a heat transfer process by statistical techniques is reported by Gallier et al. (*5*). Both a one and a two input system were studied. Record lengths from 1,500 to 5,592 data points gave results within the range of accuracy of other experimental measurements.

Goodman (*8*) has also considered the estimation of cross spectra from finite amounts of data and derived an equation for predicting the accuracy of transfer functions of one-input and one-output systems which are estimated from spectral density measurements. This equation is given in terms of the amplitude response $|H|$ and the phase response P. An approximate form (*9*) useful for calculations is given by

$$\text{Probability } \{|\hat{P} - P| < \epsilon \text{ and } |\,|\hat{H}|/|H| - 1| < \sin \epsilon\} = 1 - \left(\frac{1 - \gamma^2}{1 - \gamma^2 \cos^2 \epsilon}\right)^{k/2}$$

The number of degrees of freedom k is defined by Blackman and Tukey (*2*) and is estimated as twice the ratio of the number of data points to the number of frequencies for which the spectrum is calculated. The values $|\hat{H}|$ and $\hat{P}$ represent the estimated amplitude and phase response. γ^2 is the coherency defined as

$$\gamma^2 = \frac{|G_{xy}|^2}{G_{xx}\, G_{yy}}$$

The value of the coherency is a measure of the linear dependence between the input and output. It is equal to 1.0 if no noise extraneous to the random input signal is introduced into the sytem, and it decreases as the power of the extraneous noise increases.

The relationships between the input and output spectral densities for processes with multiple inputs have been presented by Laning and Battin (*17*) and Woodrow (*22*). These relationships may be extended directly to processes that have multiple inputs and outputs by the use of matrix notation. If there are n inputs, $x_1, x_2, \ldots, x_n$ and m outputs, $y_1, y_2, \ldots, y_m$ and if $\underline{H}$ is the matrix of the transfer functions, where H_{ij} relates output i to input j, $\underline{G}$ is the matrix of the input spectra, where G_{ii} is the power

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spectrum of input i and G_{ij} is the cross spectrum between input i and input j, $\underline{P}$ is the matrix of the output spectra, where P_{ii} is the power spectrum of output i and P_{ij} is the cross spectrum between output i and output j; then

$$\underline{P} = \underline{H}^* \underline{GH}^T$$

where $\underline{H}^*$ is the conjugate of $\underline{H}$ and $\underline{H}^T$ the transpose of $\underline{H}$. If $\underline{S}$ is the matrix of the cross spectra between input and output, where S_{ij} is the cross spectrum between input i and output j, then

$$\underline{S} = \underline{GH}^T$$

Of final interest here is the recent work of Goodman (*10*) on complex Wishart distributions. This material should enable a more coherent spectral analysis of multivariate linear systems than currently is being practiced. In addition it is important to mention the concepts of controllability and observability as introduced by Kalman (*15*). These concepts impose limitations on just what can be discovered about a system from experimental measurements and is sure to find considerable future application.

Spectral density measurements were used in the present study to illustrate that the transfer function of simulated chemical processing systems may be satisfactorily evaluated. A technique was developed such that a nonstationary random function could be used as the input to the system. The theoretical work of Goodman on predicting the accuracy of estimating the transfer function of a single-input system from spectral density measurements was tested experimentally. This experiment was carried out on a small size continuous stirred-tank reactor used in acid mixing studies. Work was also carried out to establish bounds on the accuracy of the transfer matrix of multiple input-output systems estimated from spectral density measurements. This study was made by applying nonstationary random step functions as the inputs to the linear approximations of process systems. The method of spectral density measurements for process identification is compared with other identification techniques, and its applicability to adaptive control systems is discussed.

THEORETICAL

Response Characteristics of Explicit Systems

Consider a process with inputs $x(t)$ and outputs $y(t)$ which is characterized by the linear differential equation

$$[a_n(t)D^n + a_{n-1}(t)D^{n-1} + \ldots a_0(t)]\, y(t) = [b_n(t)D^n + \ldots + b_0(t)]x(t)$$

The parameters of the process which determine the dynamic behavior are explicit in the coefficients $a_n \ldots a_0$, $b_n \ldots b_0$. If these coefficients are constants then an equivalent representation is

$$A(i\omega)\, Y(i\omega) = B(i\omega)\, X(i\omega)$$

where

$$A(i\omega) = [a_n(i\omega)^n + a_{n-1} + \ldots + a_0]$$
$$B(i\omega) = [b_n(i\omega)^n + b_{n-1} + \ldots + b_0]$$
$$Y(i\omega), X(i\omega) = \text{Fourier transforms of } y(t), x(t)$$

The ratio of the transform of the system output to input is the transfer function

$$H(i\omega) = \frac{Y(i\omega)}{X(i\omega)} = \frac{B(i\omega)}{A(i\omega)}$$

A straightforward method of determining the transfer function is to vary $x(t)$ sinusoidally at several frequencies. The amplitude ratio and phase difference between the process input and output are then the magnitude and phase angle of the transfer function

$$H(i\omega) = |H(i\omega)| \exp\,[iP(\omega)]$$

Response Characteristics of Statistical Systems

There are occasions however when it is not possible to specify the input or output of a system as an explicit function of time. Such systems become subject to study by statistical methods. Examples occur in systems where the input is of a random nature or the systems themselves generate noises which are often uncontrollable and often unmeasurable.

In the time domain the impulse response function is related to a random and stationary input and output by means of auto- and cross-covariance functions (*4, 17*). The auto-covariance function is

$$\phi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\,dt$$

and the cross-covariance function

$$\phi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\,dt$$

The relationship for a linear process with random input $x(t)$ and $y(t)$ is

$$\phi_{xy}(\tau) = \int_0^{\infty} h(\tau_1)\, \phi_{xx}(\tau - \tau_1)\,d\tau_1$$

In the frequency domain a similar relationship is obtained from the power spectral density and the cross spectrum between input and output

$$G_{xy}(i\omega) = H(i\omega) G_{xx}(i\omega)$$

where the $G(i\omega)$ are the Fourier transforms of the corresponding functions $\phi(\tau)$:

$$G_{xy}(i\omega) = \int_{-\infty}^{+\infty} \phi(\tau) e^{-i\omega\tau}\,d\tau$$

Nonstationary Noise as Process Input

The random input which was generated for use in the tests which are described later had the form of a randomized step function. For the simple on-off input used in the experimental stirred reactor test the method for deciding whether the valve should be open or closed was similar to a coin tossing test. At times $t = 0, \Delta t, 2\Delta t, \ldots$ a decision was made whether the valve should be open or closed during the next time interval, with each decision based on an equally likely chance that the valve be open or closed.

This is similar to the coin tossing experiment discussed in Laning and Battin (*17*) for which the power spectral density is

$$G_{ff}(\omega) = \frac{1 - \cos}{2\omega^2} + \frac{1}{2}\delta(\omega)$$

The $1/2\delta(\omega)$ term is due to the fact that the mean value of the random step is not zero but ½. This is a good example of nonstationary noise, as the correlation function is not independent of the origin t.

Another nonstationary random step function may also be generated. This is done by using a set of random normal deviates (*18*) and at $t = 0$ setting $f(t)$ equal to the first random deviate for the interval at $\Delta t \leqq t \leqq 2\Delta t$, etc. Thus the amplitude of the step function is constant during any interval at a value which is picked from a set of random normal deviates. The power spectral density corresponding to this type of input is

$$G_{ff}(\omega) = \Delta t \left[\frac{\sin \frac{\omega \Delta t}{2}}{\omega\Delta t/2}\right]^2$$

If such a nonstationary signal is used for the input to a process, caution must be used in determining the transfer function of the process. Consider the case in which only one sample is taken for each step length. The input may be sampled at any point during the step without changing the result of the sampling. However the output is changing continuously throughout the sampling interval, and it does make a difference when the sample is taken.

A stepped signal such as this one may be made stationary if the origin of sampling t_o is made a random variable r uniformly distributed between 0 and Δt, the length of the sampling interval. Thus a set of possible time series is obtained which is equivalent to the original random series, starting at $t_o = 0$, except shifted an amount r in time. The effect on the output is the same as if the transfer function of the process contained a pure delay term $e^{-i\omega r}$. To eliminate the effect of the time delay, so that the true transfer function is estimated, it would be necessary to average over all of the set of shifted time series. However choosing the time series which is itself the average of the set is entirely equivalent. In other words if the origin of sampling is chosen as $t_o = T/2$, where T is the length of the sampling interval, the estimated transfer function will have no delay in it owing to the nonstationary input signal. This type of sampling is demonstrated in Figure 1. Thus for one sample per step the samples are taken at the points $\frac{(2n+1)}{2}\Delta t$, for two samples per step at $\frac{(2n+1)}{4}\Delta t$, or for k samples per step $\frac{(2n+1)}{2k}\Delta t$.

ACID MIXING IN A CONTINUOUS STIRRED TANK

In order to demonstrate the application of spectral analysis to real systems the transfer function of a simple mixing process was determined. The system which was chosen for this study was a laboratory sized continuous stirred-tank reactor. A constant flow of dilute acid was fed to the vessel to maintain a steady state acid concentration in the pH range near 3. Superimposed over this flow was a small stream of 0.1N acid which was controlled by an on-off valve. This valve was used to generate a random step signal at the input of the vessel by opening and closing it according to a random sequence. The outlet acid concentration was determined by a pH meter and the results converted to acid normality. The spectral analysis was carried out on the IBM-650 with the Statisan II and III programs used.

A well-stirred vessel for acid mixing operating in an acid medium so that the effect of bicarbonate ions in the tap water is negligible has been shown (*6*) to approximate very closely the linear differential equation

$$V\frac{dN}{dt} = -(F_1 + F_2)N + F_1N_1 + F_2N_2$$

where the subscripts 1 and 2 indicate inputs 1 and 2 and F_2 is taken as the flow which is perturbed.

The transfer function for this process, forced by the disturbances on F_2, is

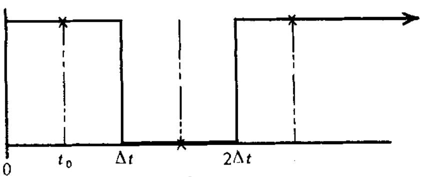
$$H(i\omega) = \frac{\dfrac{N_2 - \overline{N}}{V}}{i\omega + \dfrac{F_1 + \overline{F}_2}{V}} = \frac{b}{i\omega + a}$$

The gain is

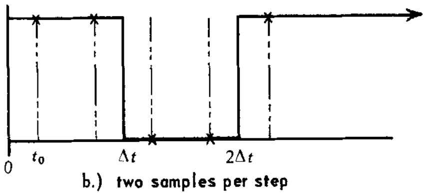
$$|H(i\omega)| = \frac{\dfrac{N_2 - \overline{N}}{F_1 + \overline{F}_2}}{\left[\left(\dfrac{\omega}{\dfrac{F_1 + \overline{F}_2}{V}}\right)^2 + 1\right]^{1/2}} = \frac{b/a}{\left[\left(\dfrac{\omega}{a}\right)^2 + 1\right]^{1/2}}$$

and the phase shift is

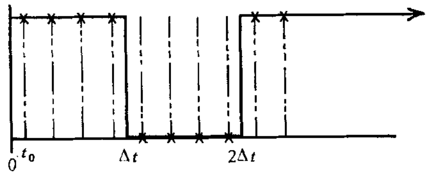
$$P(\omega) = -\arctan\frac{\omega}{\dfrac{F_1 + \overline{F}_2}{V}} = -\arctan\frac{\omega}{a}$$



a.) one sample per step



b.) two samples per step



c.) four samples per step

Fig. 1. Sampling of a random step function.

A small glass laboratory vessel of 3.06 liters was used in this experiment. It was baffled and fitted with a propeller type of stirrer so that complete mixing would occur. It was fed continuously by a stream of dilute acid F_1, prepared by mixing a stream of 1/10 N hydrochloric acid with a stream of tap water. The average value of F_1 was 1.95 liters/min.

A solenoid operated on-off valve controlled the acid stream F_2. This valve was operated by a manual switch. The acid concentration of the outlet stream was measured by a probe combination electrode and the readings converted to acid concentration in normality. It was noted that whenever the solenoid valve was opened or closed, a high frequency jump would occur in the reading of the pH meter. An electrical effect which could be conducted through the acid solution to the pH meter was very likely the cause of this jump. The effect of this jump in the pH reading was to add a pure time delay to the transfer function of the stirred tank and instrument system. This effect was indeed noted in the results of the spectral analysis determination of the transfer function.

A random input disturbance was caused by opening and closing the switch which controlled the on-off valve. Random normal deviates were obtained (*18*). At a regular interval of time Δt the next number in sequency was chosen. If it was a positive number the valve was opened, if negative the valve was closed. This gave a binary random step signal with the flow rate of F_2 controlled to be either 0.0098 liters/min. or zero flow. The magnitude of the disturbance F_2 was thus $\pm$ 0.0049, with a mean $\overline{F}_2$ of 0.0049.

For these experiments the length of the random step was chosen as 0.25 min. which gave on the average a change in valve position every 0.5 min. The time constant (or holding time) of the reactor was

$$T_c = \frac{V}{F_1 + \overline{F}_2} = 1.565 \text{ min.}$$

Thus on the average three step changes occur within the holding time of the reactor, and six samples are taken within this same time interval. The length of the run was 1 hr., which means that 240 sample points were available for analysis. All data in which the output power spectra fell below 1% of the maximum value were discarded. This was done to help insure the accuracy of the results.

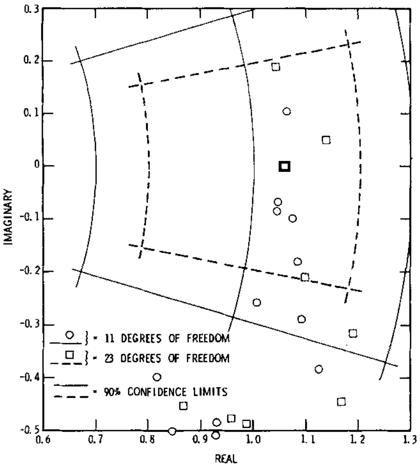


Fig. 2. Acid mixing estimates of the transfer function, uncorrected data.

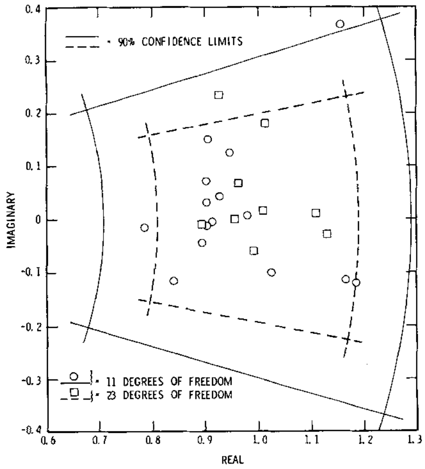


Fig. 3. Acid mixing estimates of the transfer function, corrected data.

For this system the coherency averaged to near 0.86, although it was as low as 0.63 at zero frequency. This low coherency indicates disturbances in the system, independent of that imposed on the acid flow rate, particularly slowly varying trends. These could easily be caused by fluctuations in the dilute acid-water flow rate, imperfect mixing, or instrument drift. For a coherency of 0.86 and for 11 deg. of freedom it is possible to calculate a 90% confidence that

$$\left| \frac{|\hat{H}|}{|H|} - 1 \right| < 0.290 \text{ and } |\hat{P} - P| < 16°50'$$

For 23 deg. of freedom at 90% confidence level

$$\left| \frac{|\hat{H}|}{|H|} - 1 \right| < 0.190 \text{ and } |\hat{P} - P| < 11°$$

In Figure 2 the results of $|\hat{H}|/|H|$, $\hat{P} - P$ are plotted for

$$|H| = \frac{0.050}{\left[1 + \left(\frac{\omega}{0.636}\right)^2\right]^{1/2}}$$

and

$$P = -\arctan(\omega/0.636)$$

As can be seen the data do not fall within the confidence interval, and there appears to be both a bias in the magnitude and a phase lag proportional to ωT, where T is the time constant of a pure delay. This is not entirely unexpected. The gain of the system depends greatly on the exact standardization of the pH meter. Because of the temperature adjustment which was made necessary by the difference in temperature between the standardizing buffer solution and the mixers outlet stream, about 10°C., the accuracy of the pH reading could be off as much as 0.05 pH units (at pH — 3 this represents an error of about 10%). The time lag anticipated in the effect of the solenoid valve on the pH reading was found to be $T_L = 0.105$ min., where T_L is the average taken for $\hat{P} - P/\omega$.

When the bias is removed from the data (a factor of 1.16 for the $k = 11$ system, 1.11 for the $k = 23$ system), and the time lag compensated for a pure delay of 0.105 minutes, the results fall well within the calculated confidence interval. This is shown in Figure 3.

In making a frequency response analysis of the system for the purpose of designing a control system it would not be possible to predict such biases and to correct for them. The method to be followed would be to determine the amplitude response of the system and from this the time constant of the reactor. Based on this time constant the phase response is calculated and compared with the experimentally determined phase response to find the pure time delay component.

In order to follow this procedure in this experiment it is necessary to find T_c in the equation

$$|H| = \frac{b/a}{[1 + (T_c\omega)^2]^{1/2}}$$

It is known that $b/a = 0.050$, since

$$\frac{b}{a} = \frac{N_2 - N}{F_1 - \bar{F}_2} = \frac{0.985 - 0.000674}{1.956} = 0.050$$

This is determined from the measurements of the flow rates, the input acid concentration of the on-off flow system, and the average output concentration. The equation for $|H|$ may be easily solved with a convenient plot of $|H|$ vs. $\mu = T_c\omega$ given in Gille et al. (7). The μ is found for each value of the experimental $|\hat{H}|$. From this $a = 1/T_c$ is found, where $a = \omega/\mu$. The average a is 0.721 and $T_c = 1.39$. The error in the time constant is $\frac{1.56 \times 1.39}{1.56} \times 100 = 11\%$ (see Figure 4).

Based on this time constant the phase response is determined as $\Phi = -\arctan \omega/0.721$. From this the experimental value of T_L is found as the average of $\hat{P} - \Phi/\omega$. The average T_L is 0.012. Figure 5 shows the phase response curve which was expected, based on $a = 0.636$ and the phase response curve (dotted line) for $P = -\arctan \omega/0.721 - \omega\hat{T}_L$, for $\hat{T}_L = 0.0120$. The extent of the error caused by the bias in the data is shown in Figure 6 which gives the response to a unit step function. The solid

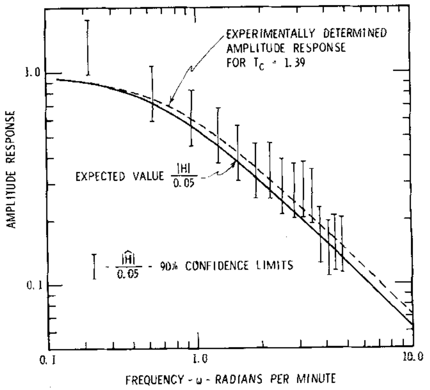


Fig. 4. Amplitude response for acid mixing experiment.

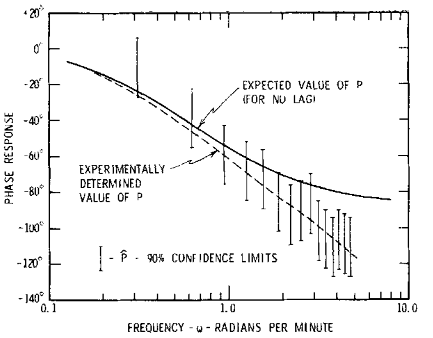


Fig. 5. Phase response for acid mixing experiment.

line represents the response for $F = 0.636$, the actual value for the stirred vessel. The dotted line represents the response curve which was determined from the experimental spectral analysis.

MULTIPLE-INPUT SYSTEMS

It is not possible to consider many dynamic chemical processes as simple one-input and one-output systems which may be described by one transfer function. Thus for any chemical system there may be several reactions taking place with several reactants and products present in the reactor, all of which are interrelated and dependent on the temperature and catalyst activity in the reactor. Such a complex system for a continuous-stirred tank reactor has been discussed in terms of matrix algebra (*14*).

It was shown that linearization around a steady state value gives the equation

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x}(t) + \underline{D}\,\underline{m}(t)$$

The transfer matrix of this system is obtained by taking the Fourier transform of the linear differential equation

$$\underline{H}(i\omega) = \underline{X}\,\underline{M}^{-1} = F[(i\omega\underline{I} - \underline{A})^{-1}\underline{D}]$$

where $\underline{X}$ and $\underline{M}$ are the Fourier transforms of $\underline{x}$ and $\underline{m}$.

To show the applicability of spectral density measurement techniques and the accuracy which may be expected from them in the estimation of the transfer matrix $H(i\omega)$, when the inputs to the system $\underline{m}(t)$ are randomly varying functions, two systems were studied. One was a second-order system in which the two process variables were concentration and temperature and the other a fifth-order system with five process variables. These systems were simulated on an IBM-650 digital computer with nonstationary random step functions used as inputs.

For multidimensional processes the spectral density relationship between input and output have been presented (*1, 17*). If P_{ij} is the spectra and cross spectra of the output H_{jn}, the transfer matrix, and G_{mk} the spectra and cross spectra of the input, then

$$P_{ij}(\omega) = \sum_{n=1}^{n}\sum_{k=1}^{n} H^{*}{}_{im}(\omega)\, H_{jk}(\omega)\, G_{nk}(\omega)$$

This can be reduced by matrix notation to

$$\underline{P} = \underline{H}^{*}\underline{G}\underline{H}^{T}$$

For the cross spectra between input and output S_{ij} the relationship is given as

$$S_{ij} = \sum_{n=1}^{n} H_{jm}\, G_{im}$$

which in matrix notation is

$$\underline{S} = \underline{G}\underline{H}^{T}$$

Thus to determine the transfer function of a multidimensional system the input spectra $\underline{G}$ and the cross-spectra input and output $\underline{S}$ must be known. The transfer matrix $\underline{H}$ is calculated as

$$\underline{H}(i\omega) = [\underline{G}^{-1}(\omega)\,\underline{S}(i\omega)]^{T}$$

A measure of the linear dependence between input and output is the coherency matrix, which is calculated as

$$\underline{\Gamma}^{2} = [(\underline{G}^{-1}\underline{S})^{T}]^{*}\underline{S}\underline{P}^{-1}$$

For an ideal case in which no error is introduced into the system which is incoherent with the input spectra $\underline{G}$, the coherency matrix is $\underline{I}$. When error is introduced into the system such that

$$\underline{P} = \underline{H}^{*}\underline{G}\underline{H}^{T} + \underline{E}$$

then the coherency matrix becomes

$$\underline{\Gamma}^{2} = [\underline{I} + (H^{*}GH^{T})^{-1}\,\underline{E}]^{-1}$$

A SECOND-ORDER SYSTEM

The first multidimensional system to be studied was the second-order system which has been discussed by Aris and Amundson (*1*) and Homan and Tierney (*12*). This is the system formed by a continuous stirred-tank reactor in which a single reaction occurs. The state of the system may be described by two quantities, the concentration of one of the reactants x and the temperature of reaction T. A mass and heat balance may be written

$$V\frac{dx}{dt} = q(x_o - x) - VR(x, T)$$

$$Vh\frac{dT}{dt} = gh(T_o - T) + (-\Delta H)\,VR(x, T) - VU^{*}(T, T_o, q_c)$$

and when made dimensionless and linearized about the mean have the form

$$\frac{dx_1}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + d_{11}m_1(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + d_{22}m_2(t)$$

In matrix notation these equations have the form shown previously.

The numerical example used in this computer study was

$$A = \begin{bmatrix} -22.730 & -10.091 \\ 21.730 & 8.663 \end{bmatrix}$$

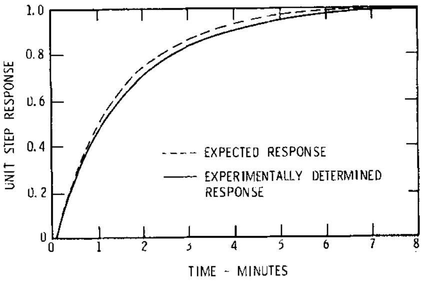


Fig. 6. Unit step response for acid mixing in perfectly mixed reactor.

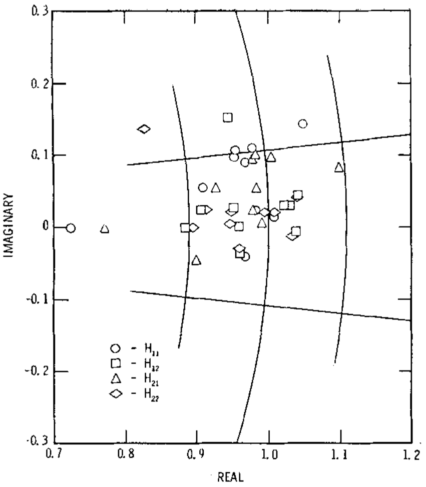


Fig. 7. Scatter diagram for two inputs perturbed and assumed to be correlated, with 80% confidence limits.

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The transfer function for this system is

$$\underline{H} = (i\omega \underline{I} - \underline{A})^{-1}$$

The linear differential equation was solved by the Runge-Kutta method for the solution of simultaneous differential equations. An integration interval of 0.01 time units was used, this short interval being chosen to make the integration accurate to the fifth decimal place.

The inputs $m_1(t)$ and $m_2(t)$ were random functions of time such that the amplitude of the function was a random number held constant for the duration of the step length. The step length was chosen as 0.4 time units with sampling carried out every 0.1 time units. Because of the nonstationary nature of this random step function it was necessary to sample at the average of the ensemble of possible steps, that is the sampling starts at 0.05 and the second sample is taken at 0.15, etc. A total of 500 samples was taken for all runs. Maximum lags of 60 and of 30 were used; this gives k deg. of freedom of 16 and 33 respectively.

One of the tests made on the second-order system consisted of perturbing both inputs $m_1(t)$ and $m_2(t)$ simultaneously. The random numbers used to generate $m_1(t)$ and $m_2(t)$ were chosen so that the two time series would be independent. However because only a finite amount of data could be used in the test there is a correlation between the two inputs. A total of 500 data points were collected, and a maximum lag of 30 was used in calculating the covariance functions. Thus the number of degrees of freedom are $k = 33$, and the resolution is $\omega = \pi/3$.

Two possibilities are open in the analysis of the results. The correlation existing between the two inputs $m_1(t)$ and $m_2(t)$ may be accounted for, or because the two are independent the correlation may be ignored. If the cross covariances g_{12} and g_{21} are taken into account, the value of the transfer matrix is obtained by direct matrix methods

$$\underline{H} = [\underline{G}^{-1}\,\underline{S}]^T$$

or explicitly

$$\underline{H} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{bmatrix} g_{22}S_{11} - g_{12}S_{21} & g_{11}S_{21} - g_{21}S_{11} \\ g_{22}S_{12} - g_{12}S_{22} & g_{11}S_{22} - g_{21}S_{12} \end{bmatrix}$$

Since the cross covariance values are available, this seems to be the advisable approach. Figure 7 presents the scatter diagram in which $|\hat{H}|/|H|$ is plotted vs. $\hat{P} - P$. The confidence bands were estimated for 80% probability by drawing in lines which exclude eight points or 20% of the data. These confidence limits are

$$|\,|\hat{H}|/|H| - 1| < .110 \text{ and } |\hat{P} - P| < 6.3°$$

which gives the estimate of the underlying coherency as approximately 0.9. The coherency matrix which was calculated from the data as

$$\Gamma^2 = [(\underline{G}^{-1}\underline{S})^T]^*\, \underline{S}\,\underline{P}^{-1}$$

is shown in Table 1. Because no error is deliberately introduced into the system it would be expected that $\Gamma^2 = \underline{I}$. Although this is not the case, it is to be noticed that the diagonal elements of this matrix are the largest and at most are near 1.0, while the off diagonal and imaginary parts of the matrix are generally smaller.

When the cross covariances g_{12} and g_{21} are not considered, then $\underline{H}$ reduces to

$$\underline{H} = \begin{bmatrix} S_{11}/g_{11} & S_{21}/g_{22} \\ S_{12}/g_{11} & S_{22}/g_{22} \end{bmatrix}$$

The results that are obtained when these straightforward calculations are carried out are shown in Figure 8. This estimation of the transfer matrix is far less accurate than was found when the full $\underline{G}$ matrix was taken into account. This is to be expected by looking at the coherencies which are calculated as

$$\gamma^2 = \frac{S_{ij}\,S^*_{ij}}{G_{ii}\,P_{jj}}$$

The averages for the ten frequencies for each of the transfer functions are

$$H_{11}, \gamma^2 = 0.40$$
$$H_{12}, \quad = 0.39$$

TABLE 1

Frequency, $3\omega/\pi$	Diagonal elements				Off-diagonal elements			
	γ_{11}^2		γ_{22}^2		γ_{12}^2		γ_{21}^2	
0	−1.4118	+0.0000i	+2.9329	+0.0000i	−0.9129	+0.0000i	+5.1067	+0.0000i
1	−0.6761	−0.9810i	+2.3335	+0.9810i	−0.6245	−0.4722i	+3.5710	+2.0274i
2	−0.2796	−0.9269i	+1.9420	+0.9296i	−0.4350	−0.4561i	+2.7439	+1.8569i
3	+0.6938	−0.2653i	+1.9121	+0.2653i	−0.0828	−0.1336i	+0.6759	+0.5058i
4	+0.8318	−0.3440i	+1.0228	+0.3440i	−0.0010	−0.1632i	+0.3980	+0.7024i
5	+0.8786	−0.2355i	+1.0592	+0.2355i	−0.0041	−0.1203i	+0.3392	+0.4159i
6	+0.9683	−0.3225i	+0.8969	+0.3225i	+0.0556	−0.1405i	+0.1190	+0.7147i
7	+0.8565	−0.1694i	+1.1322	+0.1694i	−0.0172	−0.0939i	+0.4606	+0.1987i
8	+1.0345	−0.1919i	+0.8587	+0.1919i	+0.0628	−0.0683i	−0.0344	+0.4995i
9	+0.9309	−0.0770i	+1.1077	+0.0770i	+0.0035	−0.0411i	+0.3307	+0.0094i

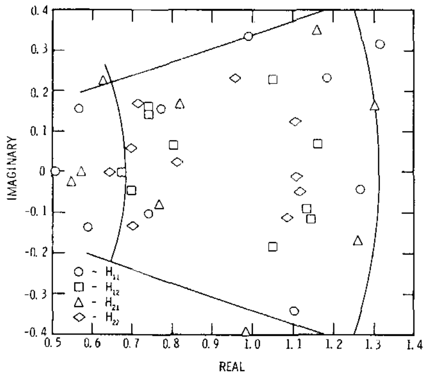


Fig. 8. Scatter diagram for two inputs perturbed and assumed uncorrelated, with 80% confidence limits.

$$H_{21}, \quad = 0.51$$
$$H_{22}, \quad = 0.53$$

If the underlying coherency between the input and output is taken as 0.50 for the four transfer function estimates, then the confidence limits for 80% probability are approximately

$$\left| \frac{|\hat{H}|}{|H|} = 1 \right| < 0.318 \text{ and } |\hat{P} - P| < 18.5^\circ$$

This is in general agreement with the results, as eleven out of forty-four of the points in Figure 8 lie outside the confidence interval.

Because the process studied was an ideal one in the sense that the equations solved were linear and no extraneous noise source existed in the process or the measurement of the inputs and outputs, only two sources of error are present. The first is the error due to sampling and the fact that aliased spectra must be used. The second is the error introduced by having only a finite amount of data.

The effect that sampling has on this system may be studied by calculating the sampled frequency response as given in the infinite series

$$TH^*(\omega) = \sum_{-\infty}^{\infty} H\left(\omega + \frac{2\pi n}{\Delta t}\right)$$

or for this second-order system

$$TH^*(\omega) = \sum_{-\infty}^{\infty} H\left(\omega + \frac{2\pi n}{0.1}\right)$$

At zero frequency the maximum error between the transfer function $H(\omega)$ and the sampled transfer function $H^*(\omega)$ is found for $H_{11}(0)$; it is a percentage error of 2.9% in the real part. At the highest frequency of interest, $\omega = 3\pi$, the maximum error is in the real part of H_{22} and is 37.2%. Although the error in the real part of $H_{22}(3\pi)$ is large in per cent, the absolute error is relatively small.

The correction for error caused by sampling is to increase the sampling frequency. However this sampling effect must be balanced by the need for resolution. Doubling the sampling frequency and maintaining the same maximum lag would mean that in the range of $\omega = 0$ to 3π there would be half as many values of the transfer function, that is instead of eighteen values in this range for an $m = 60$ there would be nine. In both cases the number of degrees of freedom are the same. The main objection occurs at low frequencies where the value of the transfer function is changing rapidly. It will be noticed that it is at these low frequencies that the greatest error occurs in the estimation of the transfer function. This error may be reduced by increasing the resolution.

It is interesting to note that by doubling the sampling frequency the largest error due to sampling for the real part of $H_{22}(3\pi)$ may be estimated as

$$R(H^*(3\pi)) = -0.01269$$

which is an error of 6.9%. Other advantages of doubling the frequency are that in the same period of sampling twice as many samples can be obtained. Thus the number of degrees of freedom may be doubled, or the maximum lag and therefore the resolution may be doubled by keeping the degrees of freedom the same. It appears that the only hindrance to increasing the frequency of sampling indefinitely, besides that of requiring better sampling instruments, is the value of the maximum lags and the length of record which can be used for calculating the covariance functions. In the case of the IBM-650 and the Statisan programs this is 60.

The other error in this system is caused by the fact that only a finite number of points can be used to determine the transfer function. This is usually accounted for in the statistical confidence limits placed on the estimated transfer function as a function of the degrees of freedom of the system and the underlying coherency between the input and output. However in this system the underlying coherency should be equal to 1.0, since no extraneous error is introduced. In fact the average estimated coherency for the first run when only one input is perturbed does equal 1.0 except at the very low frequencies. However for individual points it fluctuates from 1.01 to 0.99. The confidence limits for 80% probability at a coherency of 0.99 are

$$\left| \frac{|\hat{H}|}{|H|} - 1 \right| < 0.017 \text{ and } |\hat{P} - P| < 1^\circ$$

It appears likely that if the error due to aliasing were accounted for, the estimated transfer function would fall within these confidence limits. Thus in all cases it has been shown that the theoretical work of Goodman (*8*) has been shown to hold when treated experimentally.

FIFTH-ORDER SYSTEM

In order to test spectral density measurement techniques on larger multiple input systems a fifth-order chemical reactor system was chosen for study. The numerical example chosen was that discussed by Kalman and Koepcke (*13*) in which two simultaneous reactions take place in a continuous stirred-tank reactor:

$$A + B \xrightarrow{k_1(T)} C$$

$$2B + C \xrightarrow{k_2(T)} 2D$$

The concentrations in the reactor $A, \ldots, D$ are denoted by X_1 through X_4, and the reactor temperature is denoted by X_5. Upon linearizing about the steady state values the differential equations describing the system reduce to the usual form

$$\frac{d\underline{x}}{dt} = A\,x(t) + m(t)$$

where

$$A_{ij} = \left. \frac{\partial a_i}{\partial x_j} \right| x = x_{s,s}$$

in which

$$a_i = \frac{dx_i}{dt} = f(x_1, x_2 \ldots . x_5\, m_1, \ldots m_5)$$

Table 2. Estimated Transfer Matrix Compared with Transfer Matrix Presented as $\underline{|\hat{H}|/|H|\text{-}1}$ and $\underline{\hat{P} - P}$

$\omega = 0$									
0.1966	0.0000*I*	0.8337	0.0000*I*	0.7828	180.0000*I*	0.0000	0.0000*I*	0.4426	0.0000*I*
0.4065	0.0000*I*	1.9640	0.0000*I*	0.1907	180.0000*I*	0.0000	0.0000*I*	0.7291	−180.0000*I*
1.1501	0.0000*I*	2.1803	0.0000*I*	0.3405	0.0000*I*	0.0000	0.0000*I*	1.2291	−180.0000*I*
1.7147	180.0000*I*	2.9563	180.0000*I*	48.5151	180.0000*I*	1.2788	0.0000*I*	1.6041	180.0000*I*
0.5823	0.0000*I*	11.7729	0.0000*I*	0.4604	−180.0000*I*	0.0000	0.0000*I*	0.4925	180.0000*I*
$\omega = 5\pi/30$									
1.6597	22.1999*I*	1.5940	43.4210*I*	11.5942	−55.5910*I*	0.0000	0.0000*I*	2.8353	−89.8443*I*
0.2629	−25.3633*I*	1.1174	−5.0914*I*	17.0096	−127.3816*I*	0.0000	0.0000*I*	0.8571	165.5824*I*
1.0994	−145.6038*I*	1.2231	36.6011*I*	1.2828	−27.6611*I*	0.0000	0.0000*I*	6.1292	69.6980*I*
9.9863	−150.5471*I*	3.0490	66.5945*I*	21.5191	−76.3099*I*	1.8429	−11.4815I	4.6207	−199.2611*I*
1.1278	−46.7975*I*	0.9843	29.7552*I*	4.8043	246.7806*I*	0.0000	0.0000*I*	0.4393	−17.4377*I*
$\omega = 10\pi/30$									
0.4691	6.3267*I*	1.3724	−8.5439*I*	49.8637	−71.5837*I*	0.0000	0.0000*I*	1.7168	−75.3532*I*
0.1930	326.9406*I*	1.1939	−12.2356*I*	14.9147	185.3755*I*	0.0000	0.0000*I*	1.0813	38.4830*I*
3.4922	−208.5236*I*	2.5121	−73.8177*I*	1.0775	21.5507*I*	0.0000	0.0000*I*	3.7625	29.3698*I*
30.6031	188.7554*I*	4.2503	−85.1329*I*	73.9442	58.6876*I*	1.0058	13.7017*I*	2.4469	67.3507*I*
0.6986	131.3515*I*	1.1384	−46.1292*I*	4.0661	340.3401*I*	0.0000	0.0000*I*	1.0647	9.5026*I*
$\omega = 15\pi/30$									
1.0726	5.0980*I*	0.9444	−8.4450*I*	60.7933	127.3628*I*	0.0000	0.0000*I*	2.8749	47.8393*I*
2.0125	35.6605*I*	0.9013	10.7282*I*	35.5900	43.4147*I*	0.0000	0.0000*I*	1.9469	311.0484*I*
5.0208	28.4838*I*	1.6282	−50.0678*I*	1.3002	2.5654*I*	0.0000	0.0000*I*	4.6797	−108.9851*I*
49.8828	124.2264*I*	5.5817	−32.8634*I*	49.0892	74.1917*I*	1.0231	6.4779*I*	5.3360	−51.9948*I*
2.1380	41.0978*I*	1.4691	−3.8760*I*	11.0876	303.5571*I*	0.0000	0.0000*I*	0.7476	−23.0652*I*

The numerical values used were

$$\underline{A} = \begin{bmatrix} -0.325 & -0.5625 & 0 & 0 & 0.080 \\ -0.225 & -0.8125 & -0.014286 & 0 & -0.368 \\ 0.225 & 0.4875 & -0.107143 & 0 & 0.116 \\ 0 & 0.1500 & 0.014286 & -0.1 & 0.168 \\ 0.450 & 0.7500 & -0.035714 & 0 & -0.080 \end{bmatrix}$$

This system was simulated on the IBM-650 by solving the linear differential equation. Although the assumptions of linearity tend to somewhat remove the process from reality, it was desired to study the spectral density technique on a linear system free of the added difficulties of nonlinearities. This procedure is justified by the studies of Goodman et al. (9) who show that the transfer functions determined for nonlinear systems are affected to such an extent by the nonlinearities that they give only crude estimates. Thus this analysis is to specifically study the computational complications which arise when a high-order linear system is forced by random signals and the transfer matrix determined from spectral density measurements.

The inputs to the systems $m_1(t), \ldots, m_5(t)$ were perturbed by a randomized step function having random amplitudes held for the length of the step. The steps started at $t = 0$ and were of length 1.0 time units. The amplitude of each input $m_i(t)$ was statistically independent of the other inputs $m_j(t)$, $i \neq j$. This is also somewhat unrealistic, since the $m_i(t)$ of an actual system would consist of items such as flow rates which have a direct effect on all of the input variables. However the method of manipulation is unaffected by these independent variables, and for this linear system any effects which perturb several variables may be treated individually by superposition. The integration was performed with a modified Runge-Kutta-Gill routine and an integration interval of 0.1 units. Samples were taken every 1.0 units starting at time $t = 0.5$.

The effect which is to be expected because of the sampling is in the third decimal place at zero frequency. The effect of sampling at the highest frequency of interest $\omega = \pi/2$ is in the second decimal place for most cases.

A total of seventy-six samples was taken, and a maximum lag of 30 was used in calculating the covariance functions. This gives for the degrees of freedom

$$k = \frac{2(76 - 1/3.30)}{30} = 4.4$$

which is much lower than used in previous experiments. Based on Goodman's equation for expected results, if the underlying coherency is 0.75 for each cross spectral relationship, 80% confidence limits would be

$$\frac{|\hat{H}_{ij}|}{|H_{ij}|} - 1 < 0.600 \text{ and } (\hat{P}_{in} - P_{in}) < 36.9^\circ$$

Table 2 give the results as $|\hat{H}|/|H|$ and $\hat{P} - P$ for a number of frequencies.

Here the expected value of $|H|$ and P are used so that sampling effects are not considered. No results were obtained for $\omega = 7\pi/30$ because it was not possible to invert the $\underline{G}$ matrix for that frequency with the IBM library program used for complex matrix inversion.

The manipulation of these matrices is rather tedious on a medium speed computer, such as the IBM-650 . A total of forty-five runs on the Statisan program was required to calculate $\underline{G}$, $\underline{S}$, and $\underline{P}$ and then $[\underline{G}^{-1}\underline{S}]^T$. Use of the library program available for complex matrix inversion followed by a matrix multiplication requires about 6 min. If the coherency matrix is calculated, another 6 min. is required. The calculation of $|\hat{H}|$ and $\hat{P}$ also require about 2 min. Thus in this test where seventy-six cards and a maximum lag of 30 were used the forty-five Statisan programs required 11 min. each (including 4 min. for program loading) or 8¼ hr. The calculation of fifteen H's at

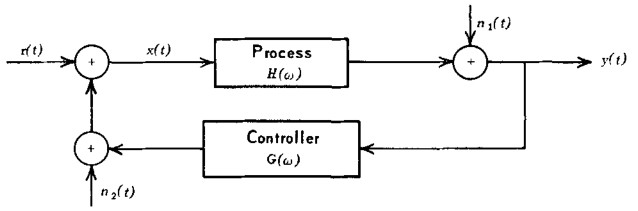


Fig. 9. Feedback control system with noise.

6 min. each requires 1½ hr. Another 1½ hr. are required for the coherency matrices and ½ hr. for the amplitude and phase response. This is a total of 11 and ¾ hr. required to obtain fifteen points on the frequency response curves of twenty-five transfer functions. Also it was not possible with the program available to invert all of the matrices that were necessary to find the transfer matrix.

The results show that in several cases the response was not at all acceptable. At zero frequency the largest amplitude response is 48.5 times as much as expected at $\pi/30$, 57.1 times as much, and the phase angles were off by as much as 295 deg. This did not improve much as higher frequencies were reached. At $\omega = \pi/2$ the largest amplitude response error was 49.9 times the expected and off by 311 deg. in phase response. It is proposed that much of this error may be eliminated by taking a much longer sample and by sampling at a higher frequency. By sampling at a higher frequency the errors caused by aliasing would be removed. A longer record length would increase the number of degrees of freedom so that the variance of the individual estimates of the transfer matrix would be reduced. However any increase in the amount of data would also increase the time of computation, which was quite large for the medium speed computer used in the present example.

POWER SPECTRAL ANALYSIS AS ONE PART OF ADAPTIVE CONTROL SYSTEMS

In the previous sections the applicability of power spectral analysis for determining the characteristics of a single multivariable process has been detailed. This was done without regard to the process control configuration used with the process. However the great utility of statistical techniques for determining process characteristics lies in the fact that they can be used during the actual operation of the system and in the presence of process noise. Thus the effect that the controllers have on the system must be considered.

One important case which has been discussed (*11, 22*) is the feedback situation in which the input to the process is influenced by the output. Figure 9 illustrates one possible feedback configuration, including the possible sources of error at the points where noise may enter the system. Woodrow has shown that for the conditions in which $r(t)$, $x(t)$, and $y(t)$ may be measured, and if the cross spectra

$$S_{n22}(\omega) = 0$$

and

$$S_{n12}(\omega) = 0$$

the best estimates for the process and controller transfer functions are

$$S_{ry}(\omega) = H(\omega) S_{rx}(\omega)$$

$$S_{rx}(\omega) = P_{rr}(\omega) + G_{ry}(\omega)$$

He also showed that for closed loop systems the best linear approximation to the process function, when only the input and output of the process is known, is given by the spectral density relationship

$$S_{xy}(\omega) = H(\omega) P_{xx}(\omega)$$

This is the same equation which has been used previously for open loop systems.

In recent years great attention has been given to adaptive control systems. In these control systems the significant characteristics of the process are automatically measured while the system is in operation. If the process characteristics change during the period of operation, the measurements will show the change; thus the control system may be altered as necessary to maintain the best possible control configuration at all times. Several examples of adaptive control systems have been discused by Truxal (*20*). One general adaptive system which he presents is shown schematically in Figure 10. In this scheme a spectral density analysis could be made to give the transfer function or transfer matrix of the process.

If the normal input to the process was a random input which was not correlated with the noise entering the process, it could be used directly for process identification. However, as pointed out by Goodman et al. (9), the process noise will generally be correlated with the process input in real processes. If this is the case, a test signal may be added as shown in the diagram of Figure 10. It is desirable that this signal be such as not to disturb the process or control, especially when the process is operating near an unstable equilibrium point, and large deviations may upset the system. A predetermined random signal of low amplitude can be applied and power spectral measurements used to determine the transfer function.

Truxal points out that in some applications it is possible that total identification is not necessary In these cases special techniques may be applied which do not require the exact analysis which the power spectral technique gives. This technique should be compared with the other methods of process identification in which the characterization is total. In the systems where noise is present this means that a statistical technique which measures either the impulse response or the frequency response is required.

The differences in computational effort between obtaining the impulse response and the frequency response are small. The additional time required to calculate the smoothed spectral estimates is a small fraction of the time required to calculate the covariance of correlation function. For resolution of the frequency spectrum into sixty parts, corresponding to a maximum lag of 60, the calculation of the smoothed spectral estimate requires an additional time of one-half that required to calculate the covariance function for 500 data points. With the same number of data points and for a maximum lag of 30 the additional time is one-fourth that of calculating the covariance. The additional calculation time may well be worth the effort, as determining the impulse response from the covariance functions requires the solution of the convolution integral. For multiple variable systems this may be a considerable task, while the spectral density functions lead directly to matrix manipulations.

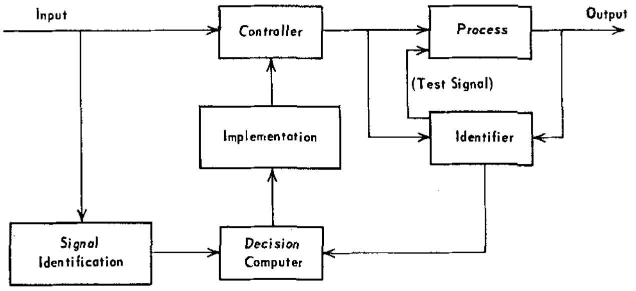


Fig. 10. An adaptive control system.

Tukey (*21*) has also pointed out that in any estimation of the statistical properties of a random signal only partial knowledge can be obtained. He states that only partial knowledge of the power spectrum $G(\omega)$ is more useful than partial knowledge of the covariance $\phi(r)$ because average values of $\phi(r)$ are hard to interpret and their statistical variances are difficult to specify and interpret.

In spite of these facts much of the work on the identification problem has been done using covariance functions to find the impulse response function (*20*). Of particular value is the work presented by Kerr (*16*) on the use of short operating records, of the order of magnitude of $2T_c$, where T_c is the settling time of the process. In the experiments done in this study the record lengths used were of the order of $40T_c$ for the stirred-tank reactor and of the order of $90T_c$ for the second-order system. Homan and Tierney (*12*) recommend a record length $T_n \geqq 100T_c$, and if the measurements are made in pieces of data, the length of each piece should be $T_p \geqq 35T_c$ and the total record length should be large. However the formulas given by Goodman (*8*) and supported by the results of this study indicate that such lengths or record may not be necessary, depending on the amount of resolution required and the extraneous noise occurring in the system.

In determining the transfer function of a process the most information is obtained for the shortest record if the extraneous noise on the system may be eliminated or its effect determined separately. This was shown in the two input system of this study. The second random input appears as noise to the first input, and although the two inputs would be uncorrelated for an infinite record, they did exhibit correlation over the finite record used. When this fact was neglected, the limits that 80% of the estimated data would lie between jumped from 11% error to 31.8% error for the amplitude response and from 6.3 to 18.5 deg. for the phase response. In this case the noise was an input which could be measured directly. If the extraneous noise cannot be measured, one possible method of eliminating it would be to perform two different spectral density measurements on the system for two different input spectra. Thus the first measurement would give

$$S_{yy}(\omega)_1 = |H|^2 S_{xx}(\omega)_1 + S_{EE}$$

and the second would be

$$S_{yy}(\omega)_2 = |H|^2 S_{xx}(\omega)_2 + S_{EE}$$

By subtracting the two results the value of the amplitude response would be

$$|H| = \sqrt{\frac{S_{yy}(\omega)_1 - S_{yy}(\omega)_2}{S_{xx}(\omega)_1 - S_{xx}(\omega)_2}}$$

One other method of analysis has been presented in which the transfer function is obtained in the presence of extraneous noise (*3*). In this technique a series of sine waves were added as the input to a process, and correlation techniques were used to eliminate the effects of the noise. Cowley reported that for each frequency of sine wave the necessary length of record was approximately $20/\omega$, although he gave no indication of the accuracy to be expected from this length of record. Suppose then that a resolution of $\Delta\omega = 1/T_c$ is desired. Ten sine waves must then be used at frequencies

$$\omega = 1/T_c, 2/T_c, \ldots, 10/T_c$$

This requires a length of record of $10T_c$ for the first frequency and so on for a total length of record for all the frequencies of $57.6T_c$. Such a length of record with a random input and power spectral analysis would give for 90% probability, and at a coherency of 0.9

$$|\hat{P} - P| < 7° \text{ and } ||\hat{H}|/|H| - 1| < 0.122$$

This would require that sampling be carried out to yield 551 samples with resolution of $\pi T_c/30$ and 36 deg. of freedom. The advantage of using a random input is in the fact that the experiment is done with only one input and does not require the shifting of frequencies on a sine wave generator from $\omega = 1/T_c$ to $\omega = 10/T_c$ as is necessary with the method presented by Cowley.

These results indicate that power spectral analysis can be used as a favorable method for characterizing process systems where total identification is necessary in the presence of extraneous noise. It may be used equally well in open or closed loop systems and in adaptive control systems. The length of calculation required compares favorably with other methods of total identification, and the results may be handled by direct matrix manipulations. For high-order systems however the calculations become quite long and require the use of very high speed computers. One other advantage of this method is that by using the theoretical equations of Goodman it is possible to plan the length of record needed and to predict the accuracy with which the process identification can be made.

CONCLUSIONS

The technique of power spectral analysis is satisfactorily used to determine the transfer functions of multiple variable process systems. A special technique of sampling is developed so that nonstationary random signals can be used as the input to the system under study. The results show that the limits of accuracy to be expected from such an analysis may be predicted by theoretical equations.

The time constant of a continuous stirred-tank reactor was experimentally determined by power spectral density analysis. For a length of record which was forty times the reactor holding time the time constant was found within 11% of the known time constant.

The transfer function of a simulated second-order system which had two random inputs was determined by a length of record ninety times the longest time constant of the system. When the correlation between the two inputs was taken into account, 80% of the results for the amplitude and phase response were within tthe limits

$$\left| \frac{|\hat{H}|}{|H|} - 1 \right| < 0.110$$

and

$$|\hat{P} - P| < 6.3°$$

However if the two inputs were assumed to be uncorrelated, as they would in fact be for an infinite record, the 80% confidence limits were found to be

$$\left| \frac{|\hat{H}|}{|H|} - 1 \right| < 0.318$$

$$|\hat{P} - P| < 18.5°$$

Both of these results were predicted by the formulas derived by Goodman.

The transfer matrix of a simulated fifth-order system was determined by a record length approximately ten times the longest time constant of the system. For this short record length of only seventy-six data points the amount of computation required was quite large, approximately 12 hr on an IBM-650 computer. Also the results for the amplitude response were in error by as much as a factor of 50, and the estimate of the phase response varied from the expected value by as much as 300 deg.

The time required to calculate each frequency estimate of the power spectral density is about one minute for 500 data points on the IBM-650 computer. This length of calculation compares favorably with other methods of total identification for processes which have extraneous noise associated with them. Because of this the application of this technique in adaptive control systems appears to be quite favorable. However as seen for the fifth-order system calculation times become quite long and impractical for high-order systems for all but very high speed computers.

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NOTATION

a = constant
A = function of $(i\omega)$
$\underline{A}$ = constant matrix
b = constant
B = function of $(i\omega)$
D = differential operator
$\underline{D}$ = constant matrix
$\underline{E}$ = error matrix
$f(t)$ = random step function
F = Fourier transform
$F_{1,2}$ = flow rates, liters/min.
G_{xx} = power spectrum
G_{xy} = cross spectrum
$\underline{G}$ = matrix of input spectra
h = impulse response function
H = transfer function
H^* = sampled transfer function
$|H|$ = amplitude response
$|\hat{H}|$ = estimated amplitude response
$\underline{H}$ = transfer matrix
$\underline{H}^*$ = complex conjugatae of transfer matrix
$\underline{H}^T$ = transpose of transfer matrix
i = imaginary root
$\underline{I}$ = identity matrix
k = degrees of freedom
$m(t)$ = forcing functions
$\underline{m}$ = vector forcing function
N = total amount of data or acid concentration, normality
$\overline{N}$ = mean acid concentration, normality
P = phase response
$\hat{P}$ = estimated phase response
$\underline{P}$ = matrix of output spectra
r = control set point
R = real part
$\underline{S}$ = matrix of cross spectra, input to output
t = time
Δt = step length in random step function
T = sampling interval
T_c = largest time constant of a system
T_L = pure time delay
T_n = total length of data collection
T_p = length of piece of data
V = volume, liters
x = variable, function of time, process input
$\underline{x}$ = vector, state function
X = Laplace or Fourier transform of x
y = variable, function of time, process output
Y = Laplace or Fourier transform of y

Greek Letters

γ^2 = coherency
Γ^2 = coherency matrix
δ = unit impulse
ϵ = error of estimation
μ = modified angular frequency
σ^2 = variance
τ = variable
ϕ_{xx} = autocovariance function
ϕ_{xy} = estimated phase response
ω = angular frequency, radians/unit time

Subscripts

c = cofunction
m = number of outputs
n = number of inputs

Superscripts

$*$ = sampled variable

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